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A duality theorem for planar three-body Ising models and their vertex model equivalences

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Abstract. A duality theorem is proved which establishes the property of self-duality in the thermodynamic limit to be a *natural* property of a large class of pure three-body Ising models.

A new form of low temperature expansion is obtained for triplet Ising models in zero field, which are closed polygon expansions and are highly lattice dependent. These closed polygon expansions readily establish an equivalence between pure triplet models and corresponding vertex models, and it is seen that a number of 'ice rules' are equivalent to triplet models at the dual point.

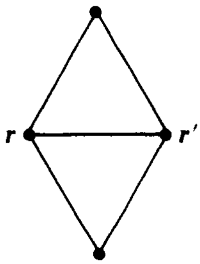
1. Introduction

Ising model systems which contain three-spin interactions have recently been considered by several authors. Some of the current interest in such models has arisen from the fact that their critical point behaviour can be quite different from the nearest-neighbour Ising model. At the present time there are two exact solutions known for triplet models (in zero field) on planar lattices which illustrate this difference in critical behaviour. Baxter and Wu (1973, 1974) have obtained the exact evaluation of the zero field partition function for the isotropic triplet model on the plane triangular lattice. Also for this model Baxter (1974, 1975) has obtained the correlation length and triplet order parameter for the nearest-neighbour model, while Baxter *et al* (1975) have conjectured the exact form of the zero field magnetization function. Sacco and Wu (1975) have also considered this triplet model as a special case of the 32-vertex model on the triangular lattice, the general solution of which is not known at the present time. Hintermann and Merlini (1972) have shown that the solution of a four-parameter anisotropic triplet model on the Union Jack lattice is equivalent to the eight-vertex model solution of Baxter (1972), thus this model possesses a variable specific heat exponent which is a function of the interaction parameters.

In the thermodynamic limit both of the above pure triplet models possess the property of self duality (Wood and Griffiths 1972, Merlini and Gruber 1972). Originally it was thought that the property of self duality in *zero field* was a highly restrictive condition (Merlini and Gruber 1972); this is in fact not the case, and in this paper we prove a duality theorem which establishes self-duality to be a *natural* property of a large class of both isotropic and anisotropic pure triplet models on planar lattices. We also show that this class of triplet models is readily transformed into equivalent vertex model problems on a variety of planar lattices. For triplet models on the Union Jack and triangular lattices Wannier's argument (Wannier 1945) for predicting the critical point is known to hold, and in each case the vertex model form of these triplet models becomes equivalent to the corresponding 'ice rules' at the *critical point* of the original triplet model. Thus in the numerical form of the weak graph series expansion for the residual entropy of square ice (Nagle 1968) the partition function is evaluated at a *singularity* of the corresponding free energy function.

2. The self-duality of planar triplet models

Consider a lattice L of classical spins σ_r ($=\pm 1$) where $\boldsymbol{r}$ labels the lattice sites. Let L be a plane triangulation in which *every* line of L connecting nearest-neighbour points $\boldsymbol{r}$ and $\boldsymbol{r}'$ is shared by two triangles in the form



(1)

and in which there are no overlapping triangles†. A pure triplet Ising model can be defined on L with a Hamiltonian in the form

$$H = -J \sum_{\Delta} \sigma\sigma'\sigma'' \qquad (J>0), \tag{2}$$

where the summation is over all triangles in L. The partition function of the model is given by

$$Z = \sum_{\{\sigma_r\}} \prod_{\Delta} \mathrm{e}^{K\sigma\sigma'\sigma''} \qquad (K=\beta J) \tag{3}$$

and can be written in the form

$$Z = (\sinh 2K)^{\frac{1}{2}T} \sum_{\{\sigma_r\}} \prod_{\Delta} \sqrt{\tfrac{1}{2}}(\mathrm{e}^{K^*} + \sigma\sigma'\sigma''\,\mathrm{e}^{-K^*}) \tag{4}$$

where T is the total number of triangles in L and K^* is the usual dual temperature defined by

$$\sinh 2K \sinh 2K^* = 1. \tag{5}$$

Consider the triangle formed by the spins σ, σ' and σ'' shown in figure 1; we restrict the number of triangles incident at each vertex of L to be even, thus these incident triangles form rings of q, q' and q'' triangles around the points σ, σ' and σ'' respectively. Following Wegner's ideas for a general transformation of variables in lattice statistics (Wegner 1973) we introduce the set of variables $\{\lambda_i\}$ where a variable λ is defined on each triangle of L by $\lambda = \sigma\sigma'\sigma''$, and consider transforming the summation in (3) to a

† In graph terminology the lattice L is a maximal plane graph.

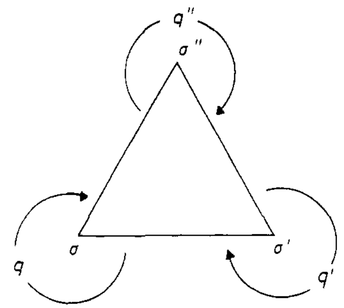


Figure 1. A triangle of spins σ, σ', and σ'' on the lattice L. Each vertex is ringed by an even number of triangles, these numbers are q, q', and q'' for the vertices of this triangle.

summation over the set $\{\lambda_j\}$. The configuration sets $\{\sigma_r\}$ and $\{\lambda_j\}$ must be coupled such that only those configurations $\{\lambda_j\}$ which correspond to spin configurations $\{\sigma_r\}$ are allowed in (3).

Consider a subgraph g of L made up entirely of triangles, and in which the number of triangles incident at any vertex of g is even. These subgraphs are precisely those graphs which contribute to the hyperbolic tangent expansion of Z (see Wood and Griffiths 1973). On any such subgraph the condition

$$\prod_{j\in g}\lambda_j=1 \tag{6}$$

must hold where the product runs over all the triangles of g. Now we observe that for the condition (6) to hold on a plane triangulation it is sufficient that the products of the q, q' and q'' λ-variables of triangles incident to the vertices σ, σ' and σ'' (see figure 1) should be equal to unity. Thus we can write

$$Z=\sum_{\{\lambda_j\}}\left\{\prod_r\phi_r\right\}\prod_j \mathrm{e}^{K\lambda_j} \tag{7}$$

where a variable ϕ_r is defined at each vertex $\boldsymbol{r}$ of L, and is unity if the above condition holds at $\boldsymbol{r}$ and is zero otherwise. Thus the general constraints ϕ_{r^*} envisaged by Wegner (1973) in this instance are such that the dual points $\boldsymbol{r}^*=\boldsymbol{r}$, and hence $L^*=L$. We can define functions $g(\lambda_j,\mu_r)$ where $\mu_r=\pm1$, and is an additional variable placed at $\boldsymbol{r}$ of L, and λ_j is one of the triangles incident to the vertex $\boldsymbol{r}$; the defining relations are

$$g(\lambda_j,1)=2^{-1/q},\qquad g(\lambda_j,-1)=\lambda_j2^{-1/q}, \tag{8}$$

where q is the number of triangles incident at $\boldsymbol{r}$. With this definition the constraints ϕ_r can be written as

$$\phi_r=\sum_{\mu_r}\prod_{i=1}^{q}g(\lambda_i,\mu_r), \tag{9}$$

and

$$Z=\sum_{\{\lambda_j\}}\prod_r\left\{\sum_{\{\mu_r\}}\prod_{i=1}^{q}g(\lambda_i,\mu_r)\right\}\prod_j\mathrm{e}^{K\lambda_j} \tag{10a}$$

$$=\sum_{\{\mu_r\}}\prod_{\Delta}\omega_{\Delta}(\mu,\mu',\mu'') \tag{10b}$$

where $\omega_\Delta(\mu, \mu', \mu'')$ is defined on each triangle of L, and is given by

$$\omega_\Delta(\mu, \mu', \mu'') = \sum_{\lambda_j=\pm 1} e^{K\lambda_j} g(\lambda_j, \mu) g(\lambda_j, \mu') g(\lambda_j, \mu'') \tag{11}$$

$$= 2^{-1/q-1/q'-1/q''}\{e^{K} + \mu\mu'\mu'' e^{-K}\}. \tag{12}$$

Thus

$$Z = \sum_{\{\mu_r\}} \prod_{\Delta} 2^{-1/q-1/q'-1/q''}\{e^{K} + \mu\mu'\mu'' e^{-K}\} \tag{13}$$

$$= 2^{\frac{1}{2}T-N} \sum_{\{\mu_r\}} \prod_{\Delta} \sqrt{\tfrac{1}{2}}\{e^{K} + \mu\mu'\mu'' e^{-K}\}. \tag{14}$$

For a triangulation defined above, on ignoring boundary effects (these are of order $1/N$) and using Euler's relation we deduce that $T = 2N$, hence combining (4) and (14) we arrive at the general duality relation

$$f(K) = \ln \sinh 2K + f(K^*) \tag{15}$$

where f is the free energy per site of *any* triplet model on a plane triangulation defined above†. Thus if Wannier's argument (Wannier 1945) holds to locate the critical point (this familiar argument may however fail in cases where the sublattices of the original lattice L are not equal), the critical point of the large class of isotropic triplet models would be the same, namely

$$e^{-2\beta_c J} = \sqrt{2} - 1 \tag{16}$$

Examples of plane triangulations are shown in figure 2.

The above development is readily generalized to pure triplet models with anisotropic triplet fields with K^* and K in (4) and (14) replaced by K_α^* and K_α respectively where K_α is the interaction constant of a triangle or a set of triangles in L. Thus in the case of the Union Jack triangulation Hintermann and Merlini showed that for four interaction constants

$$f(K_1, K_2, K_3, K_4) = \tfrac{1}{4} \sum_{\alpha=1}^{4} \ln \sinh 2K_\alpha + f(K^*_{\pi(1)}, K^*_{\pi(2)}, K^*_{\pi(3)}, K^*_{\pi(4)}) \tag{17}$$

which can also be derived from this generalization of (15). In (17) π is a permutation belonging to the dihedral group of order 4. Generalized duality relations of this type can be derived for a large class of triplet models using the development leading to (14). Thus the anisotropic version of the triangular lattice triplet model (see § 3) which has six interaction constants $K_1, \ldots, K_6$ on triangles surrounding the points of a triangular sublattice, satisfies a corresponding duality relation to (17) with π now the dihedral group of order 6.

† This result can also be established in terms of the group theoretical treatment of duality relations as developed by Merlini and Gruber (1972). In their notation a choice of the identity map $d\boldsymbol{B} = \boldsymbol{B}^* \equiv \boldsymbol{B}$ (for all $\boldsymbol{B}$ in $\mathbb{B}$) for the bijection d: $\mathbb{B} \to \mathbb{B}^*$ implies $K \equiv \Gamma^*$ and the model is self dual by definition of self duality.

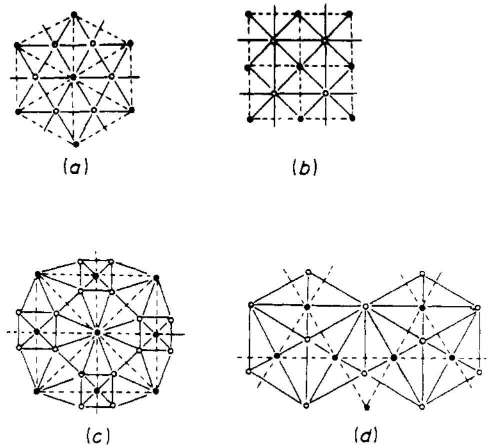
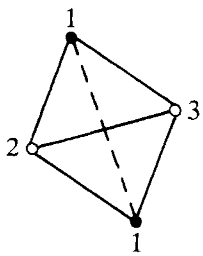


Figure 2. Some examples of plane triangulations: (*a*) the regular triangular lattice; (*b*) the Union Jack lattice; (*c*) a triangulation of the bathroom tile lattice; and (*d*) a triangulation of the diced lattice. Each lattice is divided into sublattices L_1 and L_{23} denoted by the full and open circles respectively. The lattice L_1 is the dual lattice to L_{23} and is shown by the broken lines, the dual lattices shown in (*a*), (*b*), (*c*), and (*d*) respectively are the triangular lattice, the square lattice, the Union Jack lattice, and the Kagomé lattice.

3. Vertex model equivalences of planar triplet models

Triplet models on a number of plane triangulations have interesting equivalences with vertex model problems, thus Hintermann and Merlini (1972) introduced the Union Jack triangulation via its correspondence with the eight-vertex model of Baxter (1972). We take the same definition of a plane triangulation as given in § 2, and consider a triangulation in the form



(18)

where the sublattice L_1 (full circles) is the dual lattice of the sublattice L_{23} (open circles). Examples of (18) where the dual lattice is the triangular, Kagomé and Union Jack lattice are shown in figure 2 by the broken lines.

Consider again an isotropic triplet model Hamiltonian defined on such a triangulation, then following Baxter and Wu (1974) we can write this Hamiltonian in the form

$$H = \sum^{23} \sigma_{L_1}(\lambda_1 + \lambda_2 + \ldots + \lambda_q) \qquad (19)$$

where the summation is over the nearest-neighbour bonds of the lattice L_{23}, and

$\lambda = \sigma_{L_2}\sigma_{L_3}$, with σ_{L_α} being the spin variables on the sublattice L_α. In (19) q is the number of triangles ringing the lattice point σ_{L_1}, again we assume that q is even but not necessarily the same at each point of L_1. Following Wegner (1973) the partition function can now be written in the form

$$Z = \sum_{\{\sigma_{L_1}\}} \sum_{\{\lambda\}} \prod^{23} \{\exp[K\sigma_{L_1}(\lambda_1+\lambda_2+\ldots+\lambda_q)]\} \tfrac{1}{2}(1+\lambda_1\lambda_2\ldots\lambda_q) \tag{20}$$

where Wegner's constraints are

$$\phi_{r^*} = \sum_{\mu_{r^*}} \prod_{i=1}^{q} g(\lambda_i, \mu_{r^*}) = \tfrac{1}{2}(1+\lambda_1\lambda_2\ldots\lambda_q) \tag{21}$$

and where the g-functions are again defined by (8), but now r^* are the points of the L_1 sublattice. Following the procedure of § 2 (see also Baxter and Wu 1974) we can eliminate the λ-variables in (20) (these are now products of *pairs* of spins) and the partition function can be expressed in the form

$$Z = \sum_{\{\sigma_{L_1}\}} \sum_{\{\mu_{L_1}\}} \prod 2^{-1/q-1/q'} (e^{K(\sigma_{L_1}+\sigma'_{L_1})} + \mu_{L_1}\mu'_{L_1}\, e^{-K(\sigma_{L_1}+\sigma'_{L_1})}) \tag{22}$$

where the product is over all nearest-neighbour pairs of points $(\mu_{L_1}, \sigma_{L_1})$ and $(\mu'_{L_1}, \sigma'_{L_1})$ on the L_1 lattice at which are incident q and q' triangles respectively. The modification to (22) caused by anisotropic triplet fields where triangles sharing a common bond of L_{23} and incident at neighbouring points $(\sigma_{L_1}, \mu_{L_1})$, $(\sigma'_{L_1}, \mu'_{L_1})$ have interactions K and K' respectively, is given by

$$Z = \sum_{\{\sigma_{L_1}\}} \sum_{\{\mu_{L_1}\}} \prod 2^{-1/q-1/q'} (e^{K\sigma_{L_1}+K'\sigma'_{L_1}} + \mu_{L_1}\mu'_{L_1}\, e^{-K\sigma_{L_1}-K'\sigma'_{L_1}}). \tag{23}$$

It is possible to derive a *low temperature* expansion of the forms (22) and (23) in terms of only the weak polygon subgraphs of the lattice L_1. As far as the authors are aware these are new forms of low temperature expansions in which the expansion variables turn out to be highly dependent upon lattice structure. The expansions are a class of weak graph expansions but seem to arise from a more complicated form of the partition function than envisaged by Nagle (1968) in a general treatment of weak graph expansions. For an isotropic triplet model we can write (22) in the form

$$Z = 2^{-N_1} \sum_{\{\sigma_{L_1}\}} \sum_{\{\mu_1\}} \prod [(c+\sigma_{L_1}s)(c+\sigma'_{L_1}s) + \mu_{L_1}\mu'_{L_1}(c-\sigma_{L_1}s)(c-\sigma'_{L_1}s)] \tag{24}$$

where $c = \cosh K$, and $s = \sinh K$. On expanding the product in (24) we can denote each product pair $\mu_{L_1}\mu'_{L_1}$ by a line joining nearest -neighbour points of L_1, and on summing over the variables $\{\mu_{L_1}\}$ in the usual way only the weak polygon embeddings (these are subgraphs of L_1 where the degree of each vertex is even) will survive in the expansion; this is exactly the same phenomena as occurs in the hyperbolic tangent expansion of the zero field Ising model (Domb 1974). Thus each term in the expansion of (24) is now associated with a polygon subgraph $g(p)$ of L_1, and consists of a product of $2s$ factors where s is the number of lines of L_1. Thus we can write

$$Z = \sum_{\{\sigma_{L_1}\}} \sum_{g(p)} \prod_{ij \in L_1} \tau_{ij}, \tag{25}$$

where the product is over all the edges of L_1 and

$$\tau_{ij} = (c - \sigma_i s)(c - \sigma_j s) \tag{26}$$

if ij is an edge of the graph $g(p)$, and

$$\tau_{ij} = (c + \sigma_i s)(c + \sigma_j s) \tag{27}$$

if ij is not an edge of $g(p)$. Since every edge of L_1 is represented by a product of *two* factors in (25) we can rearrange the expansion in the form of a vertex expansion and write

$$Z = \sum_{g(p)} \prod_{i=1}^{N_1} V(i) \tag{28}$$

where $V(i)$ is the weight of the ith vertex of L_1, and is given by

$$V(i) = \sum_{\sigma_i} (c - \sigma_i s)^{l_i} (c + \sigma_i s)^{q - l_i} = \omega(l_i). \tag{29}$$

In (29) q is the degree of the ith vertex and l_i is the number of lines of the graph $g(p)$ incident to the ith-vertex of L_1 (q and l_i are both even), thus

$$\omega(l_i) = \omega(q - l_i) = (c - s)^{l_i}(c + s)^{q - l_i} + (c + s)^{l_i}(c - s)^{q - l_i} \tag{30}$$

and we can write Z in the form

$$Z = \left(\prod_q \omega(0)^{N_q}\right) \sum_{g(p)} \prod_q \left(\frac{\omega(2)}{\omega(0)}\right)^{n_2 + n_{q-2}} \left(\frac{\omega(4)}{\omega(0)}\right)^{n_4 + n_{q-4}} \cdots \tag{31}$$

where n_α is the number of vertices of degree α in $g(p)$ at vertices of degree q in L_1 (q may vary over the points of L_1), with N_q being the number of points of L_1 with degree q. We can now consider a few examples of (31).

3.1. *The Union Jack lattice*

For this lattice L_1 is the simple quadratic lattice (see figure 2), thus setting $q = 4$ in (31) we obtain

$$Z = \omega(0)^N \sum_{g(p)} [g(p)] \left(\frac{\omega(2)}{\omega(0)}\right)^{n_2} \tag{32}$$

where $[g(p)]$ is the number of embeddings of the closed polygons $g(p)$ on L_1, and n_2 is the number of vertices of degree 2 in $g(p)$. The expansion parameter is

$$\frac{\omega(2)}{\omega(0)} = 2(1 - v^2)^2 / [(1 + v)^4 + (1 - v)^4], \tag{33}$$

with $v = \tanh K$, and

$$\omega(0) = \mathrm{e}^{4K} + \mathrm{e}^{-4K}. \tag{34}$$

Clearly the expansion (32) is related to the generating function used to obtain the residual entropy of square ice (Nagle 1968, 1974). It is known that the critical point of this triplet model is given by (16), which yields $\omega(2)/\omega(0)=\frac{1}{3}$ at the critical point, thus at the critical point the partition function becomes *identical* with Nagle's entropy expansion

$$Z_c=4^N(\tfrac{3}{2})^N \sum_{g(p)} [g(p)](\tfrac{1}{3})^{n_2}. \tag{35}$$

Hence the point $x=\frac{1}{3}$ of Nagle's expansion (Nagle 1974) is a singular point of the free energy function.

3.2. *Triplet model on the triangular lattice*

A correspondence between the ice rule on the triangular lattice and the pure triplet model on this lattice at its critical point (again given by (16)) was noted by Baxter and Wu (1974) in their equivalent colouring problem for this triplet model. This equivalence can also be seen from the expansion (31) where on putting $q=6$ at all sites of L_1

$$Z=\omega(0)^N \sum_{g(p)} [g(p)]\left(\frac{\omega(2)}{\omega(0)}\right)^{n_2+n_4}, \tag{36}$$

where

$$\omega(0)=e^{6K}+e^{-6K}, \tag{37}$$

and

$$\omega(2)/\omega(0)=u/(u^2-u+1) \qquad (u=e^{-4K}). \tag{38}$$

Thus at the critical point (16)

$$Z_c=(10\surd 2)^N \sum_{g(p)} [g(p)](\tfrac{1}{5})^{n_2+n_4} \tag{39}$$

which apart from a numerical factor is Nagle's weak graph expansion for triangular ice (Nagle 1968). Although the ice rule restricts the number of vertex weights to be 6 and 20 respectively for the square and triangular lattices, both (35) and (39) can be derived using the weak graph transformation (Wu 1969) on the corresponding 8- and 32-vertex models.

3.3 *The diced lattice*

A triplet model on the diced lattice (see figure 2) has a weak graph expansion identical to (32) on the Kagomé lattice formed by the L_1 sites. It can readily be shown that the residual entropy series for Kagomé ice is also given by the expansion (35) (apart from a numerical factor) with $g(p)$ defined over the sites of the Kagomé lattice. The residual entropy of Kagomé ice is as yet unsolved (Lin 1975a, b), however another ice rule equivalence will exist at the dual point (16), although this may turn out not to be a critical point.

3.4. *The bathroom tile lattice*

A triplet model can be defined over the triangulation of the bathroom tile lattice shown in figure 2. Here the L_1 sites form a Union Jack lattice with $N_4=N_8=N$. Thus the low temperature closed polygon expansion (31) takes the form

$$Z=\omega(0,4)^N\omega(0,8)^N\sum_{g(p)}\left(\frac{\omega(2,4)}{\omega(0,4)}\right)^{n_2}\left(\frac{\omega(2,8)}{\omega(0,8)}\right)^{n_2+n_6}\left(\frac{\omega(4,8)}{\omega(0,8)}\right)^{n_4}[g(p)] \tag{40}$$

where $\omega(\alpha,q)$ is used to denote $\omega(q-\alpha)$ of (30), thus the index n_2 on the first parenthesis in (40) is the number of degree 2 vertices of $g(p)$ at the $q=4$ points of L_1. The partition function (40) corresponds to a special case of the vertex model problem on the Union Jack lattice which has the usual arrow reversal symmetry. Thus at $q=8$ sites 128 vertex configurations are allowed while at the $q=4$ sites the usual 8-vertex configurations are allowed.

The above vertex model equivalence is a special case of a general equivalence between isotropic pure triplet models on plane triangulations and vertex models on the L_1 lattice. The vertex models all possess arrow reversal symmetry where only even numbers of arrows point towards any vertex of L_1. In bond language for the vertex configurations (see Lieb and Wu 1972) this means that a vertex configuration of L_1 has zero or an even number of bonds incident to the vertices of L_1, thus a summation over vertex configurations is precisely the summations over the closed polygons in (31) on L_1. The vertex weights of this equivalent vertex model are defined by (30) which can be written

$$\omega(l_i)=\omega(q-l_i)=2\cosh[(q-2l_i)K]. \tag{41}$$

Thus for the Union Jack triplet model the equivalent vertex model is the 8-vertex model (L_1 = the square lattice), and the special case of the vertex weights is

$$\omega(0)=\omega(4)=2\cosh 4K \tag{42}$$

and the remaining six degree 2 vertices have weights $\omega(2)=2$. On the triangular triplet model (Sacco and Wu 1975)

$$\omega(0)=\omega(6)=2\cosh 6K \tag{43}$$

while the remaining thirty degree 2 and 4 vertices have weights

$$\omega(2)=\omega(4)=2\cosh 2K. \tag{44}$$

For triplet models of the types shown in figure 2 with anisotropic triplet fields the corresponding vertex models on the L_1 lattice are easily derived. Thus if a degree q-vertex of L_1 has the q interaction parameters $K_1, K_2, \dots, K_q$ on the q triangles ringing the vertex then the $2^{q/2}$ vertex weights of this vertex of L_1 are given by

$$V_\alpha=2\cosh\left(\sum_{i=1}^{q}(-1)^bK_i\right)\qquad(\alpha=1,2,\dots,2^{q/2}) \tag{45}$$

where $b=1$ if the 'triangle' K_i contains an edge (actually a half edge, see (18)) of L_1 which is covered by a bond of the vertex configuration otherwise $b=2$.

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